

THE FRACTIONAL LONDON EQUATION AND THE FRACTIONAL PIPPARD MODEL FOR SUPERCONDUCTORS

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Abstract

With the discovery of new superconductors there was a running to find the justifications for the new properties found in these materials. In order to describe these new effects some theories were adapted and some others have been tried. In this work we present an application of the fractional calculus to study the superconductor in the context of London theory. Here we investigated the linear London equation modified by fractional derivatives for non-differentiable functions, instead of integer ones, in a coarse grained scenario. We apply the fractional approach based in the modified Riemann-Liouville sense to improve the model in order to include possible non-local interactions and the media. It is argued that the effects of non-locality and long memory, intrinsic to the formalism of the fractional calculus, are relevant to achieving a satisfactory phenomenological description. In order to compare the present results with the usual London theory, we calculated the magnetic field distribution for a mesoscopic superconductor system. Also, a fractional Pippard-like model is proposed to take into account the non-locality beside effects of interactions and the media. We propose that α parameter of fractionality can than be used to create an alternative way to characterize superconductors.

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I. INTRODUCTION

The non-integer order calculus, also called or fractional calculus (FC) has become an promising emergent tool for scientific and applied research. A landscape of various different fields have already been touched by this powerful alternative for modeling. Thinking in terms of complex systems, there exist studies that make uses of these for modeling physical, biological, human and social systems, among other. In the scope of several research fields, including physics, efforts for the understanding connections between the dynamics and complex systems have been found in the scientific literature, specially in the non local theories. Among the motivations, one is that the use of these theories may yield a much more elegant and effective treatment of problems in science. The FC provides us with a set of mathematical tools to generalize the concept of derivative and integral operators with integer order to their respective extensions of an arbitrary real order [1]. Nonlocal theories and memory effects have been thought as connected to complexity and admit a treatment in terms of FC [2, 3]. In this context, the nondifferentiable nature of the microscopic dynamics may be connected with time scales so as to approach questions in the realm of complex systems [4].

In a coarse grained scenario, it was recently proposed a simple alternative definition to the Riemann-Liouville derivative [6, 7], called modified Riemann-Liouville (MRL) fractional derivative, that has the advantages of both the standard Riemann-Liouville and Caputo fractional derivatives: it is defined for arbitrary continuous (non differentiable) functions and the fractional derivative of a constant is equal to zero. This kind of fractional calculus approach seems to give a mathematical framework for dealing with dynamical systems defined in coarse-grained spaces and with coarse-grained time and, to this end, to use the fact that fractional calculus appears to be intimately related to fractal and self-similar functions. We would like to stress that the choice of MRL approach, besides the points already mentioned, is justified by the fact that chain and Leibniz rules acquires a simpler form, which helps a great deal if changes of coordinates are performed. Moreover, causality seems to be more easily obeyed in a field-theoretical construction if we adopt this approach.

The non-differentiability and randomness are mutually related in their nature, in such a way that studies in fractals on the one hand and fractional Brownian motion on the other hand are often parallel [6]. A function which is continuous everywhere but not always differentiable necessarily exhibits random-like or pseudo-random-features, in the sense that

various samplings of this functions on the same given interval will be different. This may explain the huge amount of literature which extends the theory of stochastic differential equation to stochastic dynamics driven by fractional Brownian motion. The most natural and direct way to question the classical framework of physics is to remark that in the space of our real world, the generic point is not infinitely small (or thin) but rather has a thickness. A coarse-grained space is a space in which the generic point is not infinitely thin, but rather has a thickness; and here this feature is modeled as a space in which the generic increment is not dx , but rather $(dx)^\alpha$ and likewise for coarse grained with respect to the time variable t .

In this paper we have worked out the linear London equation [5] modified by fractional derivatives for non-differentiable functions [6, 7], instead of integer ones, in a coarse grained scenario. We also proposed a fractional Pippard model for superconductors. We applied the fractional approach in the MRL sense to improve the model in order to include possible non-local interactions and the influence of the media. It is argued that the effects of non-locality and long memory, intrinsic to the formalism of the fractional calculus, are relevant to achieving a satisfactory phenomenological description. By considering fractional derivatives in space, a generalized fractional Laplacian is introduced and by means of a transformation of variables called complex transform [8] we construct and solve a fractional London equation. Also, based on a Chambers model and following the Dressel and Gruner's book [12], we proposed a fractional Pippard model for superconductors.

This paper is outlined as follows: In section II, we give some background about the MRL formalism, In section III we develop the fractional London equation with the solution. In section IV is devoted to review some aspects of the Chambers development for Chambers formula. In section V we present the fractional Pippard-like model based on Chambers formula and in section V we cast our the concluding comments and prospects for further investigation.

II. MODIFIED RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVE

The well-tested definitions for fractional derivatives, so called Riemann-Liouville and Caputo have been frequently used for several applications in scientific periodic journals. In spite of its usefulness they have some dangerous pitfalls. Recently, it was proposed the MRL

definition for fractional derivative [6, 7], and its basic definition is given by

$$\begin{aligned}
D^\alpha f(x) &= \lim_{h \rightarrow 0} h^{-\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x + (\alpha - k)h) = \\
&= \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dx} \int_0^x (x - t)^{-\alpha} (f(t) - f(0)) dt; \\
0 &< \alpha < 1.
\end{aligned} \tag{1}$$

Some advantages can be cited, first of all, using the MRL definition we found that derivative of constant is zero, and second, we can use it so much for differentiable as non differentiable functions. They are cast as follows:

(i) Simple rules:

$$\begin{aligned}
D^\alpha K &= 0, \\
Dx^\gamma &= \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + 1 - \alpha)} x^{\gamma - \alpha}, \quad \gamma > 0, \\
(u(x)v(x))^{(\alpha)} &= u^{(\alpha)}(x)v(x) + u(x)v^{(\alpha)}(x).
\end{aligned} \tag{2}$$

(ii) Simple Chain Rules:

$$\frac{d^\alpha}{dx^\alpha} f[u(x)] = \frac{d^\alpha f}{du^\alpha} \left(\frac{du}{dx} \right)^\alpha, \tag{3}$$

for non differentiable functions and

$$\frac{d^\alpha}{dx^\alpha} f[u(x)] = \frac{df}{du} \frac{d^\alpha u}{dx^\alpha}, \tag{4}$$

for coarse-grained space.

Details of the formalism can be found in the cited references and references therein.

Now that we have set up these fundamental expressions, we are ready to carry out the calculations of main interest, the construction and the solutions to our fractional London equation and a development of a fractional Pippard model.

III. LINEAR FRACTIONAL LONDON EQUATIONS

For investigating the magnetic-field distribution feature in the fractional formalism, we need the modified London equation. For comparison with that in the integer case we will

first review the derivation of the modified London equation which is usually written as [5]:

$$\mathbf{B} + \lambda^2 \nabla \times (\nabla \times \mathbf{B}) = 0. \quad (5)$$

where λ is the London penetration depth and \mathbf{B} is the Magnetic induction vector field.

with the standard form [11],

$$\lambda^2 \nabla^2 \mathbf{B}(r) - \mathbf{B}(r) = -\Phi_0 \delta^{(2)}(\mathbf{r}) \hat{z}, \quad (6)$$

Here $\Phi_0 = 2\pi\hbar/e^*$ is the quantum flux and ∇^2 is the vectorial Laplacian. The magnetic field is in \hat{z} direction and depends only on radial coordinates r .

This equation have a well known exact solution given by:

$$B = \frac{\Phi_0}{2\pi\lambda} K_0\left(\frac{r}{\lambda}\right), \quad (7)$$

where K_0 is the zero order Hankel function.

With this 2-D geometry we can write the operator ∇^2 as

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{d}{dr} \right), \quad (8)$$

letting to a rewrite eq. (6) as

$$-\frac{\lambda^2}{r} \frac{d}{dr} \left(r \frac{dB}{dr} \right) + B(r) = \Phi_0 \delta^{(2)}(\mathbf{r} - \mathbf{r}_0). \quad (9)$$

In order to rewrite the equation above with fractional derivatives we use a generalized fractional Laplacian, in the MRL's sense, of a form [9, 10] given by

$$\frac{1}{r^\alpha} D_r^\alpha (r^\alpha D_r^\alpha B(r)) \quad . \quad (10)$$

with this fractional operator we can write the fractional London equations as

$$\frac{\lambda^{2\alpha}}{r^\alpha} D_r^\alpha (r^\alpha D_r^\alpha B(r)) + B(r) = \Phi_0 \delta^{(2)}(\mathbf{r} - \mathbf{r}_0) \quad . \quad (11)$$

Now, to solve eq. (11) we apply a change of variable called fractional complex transform [8]. The fractional complex transform is used to convert fractional differential equations with modified Riemann-Liouville fractional derivatives into ordinary differential equations,

so that all analytical methods devoted to advanced calculus can be easily applied to fractional calculus.

To proceed, we define the new variable

$$R = \frac{pr^\alpha}{\Gamma(1 + \alpha)}. \quad (12)$$

With the help of the above transformation and the rule given by 4, we obtain an integer one of the similar form of eq.(9) but with argument depending on variable R . and with known solution. The final form of solution of the eq. (11) is given by

$$B(r/\lambda) = \frac{\Phi_0}{2\pi\lambda^\alpha} K_0 \left[\frac{(r/\lambda)^\alpha}{\Gamma(\alpha + 1)} \right] \quad (13)$$

The above expression can be used to compare the behavior of this solution to the well known integer one by the use of a algebraic or graphic computational software. Some comments will be given in the concluding remarks.

Local electrodynamics assumes that the response at one point in the material only depends on the electric field at this point. This assumption breaks down in the case of metals with a long mean free path and in the case of superconductors with a long coherence length ξ_0 . The anomalous skin effects become important if $\delta < l$. For superconductors the London limit is exceeded if $\lambda < \xi_0$. In both cases the influence of electrons, which feel a different electric field at some distant point, becomes important. To take into account the non-local effects beside the interactions and media measured by a fractionality parameter, we propose a fractional Pippard-like model.

IV. THE CHAMBERS FORMULA AND THE PIPPARD'S MODEL

To work out a Pippard-like fractional model, we proceed as in [12] and we give here the details for clarity in the procedure. The Chamber model is used in the theory anomalous skin effect for the treatment of the non-local conductivity. The anomalous skin effect is the name given to the behavior of a metal under such conditions of purity and low temperature that the high-frequency oscillations of electric field and current are confined within a surface layer of thickness much less than the mean free path [13]. In this model, the current density in a normal metal in which the electric field varies over a mean free path is determined. For this,

consider the \vec{q} moment dependent response is intimately related to the non-local conduction where the current density \vec{J} at the position \vec{r} is determined also by fields at other locations $\vec{r} \neq \vec{r}'$. Following, an approximate expression for the current density which depends on the spatial distribution of the applied electric field is developed for a situation that may occur in the case of clean metals at low temperatures when the mean free path is large. Considering in the sequence an electron at \vec{r} and subjected to an electric field $\vec{E}(\vec{r})$, moving to another position taken to be the origin of the coordinate system, where the field is different from that at initial position. However, because of collisions with the lattice or impurities, the momentum acquired by the electron from the field at \vec{r} decays exponentially as the origin is approached. This last consideration will be relaxed into the fractional approach. The characteristic decay length defines the mean free path, and the currents at the origin are the result of the fields $\vec{E}(\vec{r})$ within the radius of $l = v_F \tau$. The non-local response follows the argumentation that when an electron moves from a position $(\vec{r} - d\vec{r})$ to \vec{r} , it is influenced by an effective field $\vec{E}(\vec{r}) \exp(-r/l)$ for a time $d\vec{r}/v_F$. The momentum acquired $d\vec{p}(0)$ in the direction of motion and at initial position is

$$d\vec{p}(0) = \frac{e d\vec{r} \vec{r}}{v_F r} \cdot \vec{E}(\vec{r}) \exp(-r/l); \quad (14)$$

By integrating the eq. (14) from the origin to infinity, the total change in momentum for an electron at the origin is found. Performing this calculation for all directions allows to map out the momentum surface in a non-uniform field, and the deviations from a sphere centered at the origin constitute a current density \vec{J} . The following arguments lead to the expression of this current density. Only electrons residing in regions of momentum space not normally occupied when the applied field is zero contribute to the current density.

The density of electrons ΔN moving in a solid angle $d\Omega$ and occupying the net displaced volume in momentum space ΔP is

$$\Delta N = \frac{\Delta P}{P} N = \frac{(mv_F)^2 d\Omega dp}{\frac{4}{3}\pi (mv_F)^3} N = \frac{3N d\Omega dp}{4\pi v_F m}, \quad (15)$$

where P is the total momentum space volume. The contribution to the current density from these electrons is

$$d\vec{J} = -\Delta N e v_F \frac{\vec{r}}{r} = -\frac{3Ne}{4\pi m} \frac{\vec{r}}{r} d\Omega dp. \quad (16)$$

Substituting eq. (14) into this equation and integrating over the currents given above yields

$$\vec{J}(\vec{r}=0) = \frac{3\sigma_{dc}}{4\pi l} \int \frac{\vec{r} \left[\vec{r} \cdot \vec{E}(r) \exp(-r/l) \right]}{r^4} d\vec{r}, \quad (17)$$

since a volume element in real space is $r^2 dr d\Omega$ and $\sigma_{dc} = Ne^2\tau/m = Ne^2l/(mv_F)$. Equation (17) represents the non-local generalization of Ohm's law for free electrons, and reduces to $\vec{J} = \sigma_{dc}\vec{E}$ for the special case where $l \rightarrow 0$, as expected. The Chambers formula is valid for finite momentum, but as the Fermi momentum is not explicitly included its use is restricted to $\vec{q} < \vec{k}_F$, and in general to the small \vec{q} limit.

Note that, in view of $\sigma_{dc} = Ne^2\tau/m = Ne^2v_F l/m$, the mean free path drops out from the factor in front of the integral. For superconductors, the vector potential \vec{A} is proportional to \vec{J} , and with $\vec{E} = i(\omega/c)\vec{A}$ we can write accordingly

$$\vec{J}(0) \propto \int \frac{\vec{r} \left[\vec{r} \cdot \vec{A}(r, \omega) \right]}{r^4} F(r) d\vec{r}. \quad (18)$$

With this background, we can now proceed to the obtainment of the fractional Pippard-like model.

V. THE FRACTIONAL PIPPARD-LIKE MODEL

Here, we will assume that the kernel function $F(r)$ have to take into account that the charge carriers can be thought as pseudo-particles that carries the implicit information of an effective field and the media, attributing to each location a probability $\frac{(r-r')^{\alpha-1}}{\Gamma(\alpha)}$. Another possibility is to attribute to $F(r)$ an stretched exponential, that is, the two parameters Mittag-Leffler function as the probability factor that substitute the exponential in the Pippard's original formula.

The formula is than

$$\vec{J}(\vec{r}) \propto \int \frac{\vec{r} \left[\vec{r} \cdot \vec{A}(r, \omega) \right]}{r^4} F(r) E_{\alpha\beta}(-r/l) d\vec{r} \quad (19)$$

where $F(r) = \frac{(r-r')^{\alpha-1}}{\Gamma(\alpha)}$, ξ is the correlation length and the two parameter Mittag-Leffler functions is given by $E_{\alpha\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha x + \beta)}$. With the $F(r)$, eq. (19) is now

$$\vec{J}(\vec{r}') \propto \frac{1}{\Gamma(\alpha)} \int (r - r')^{\alpha-1} K(A, r, \xi) d\vec{r} = I^\alpha K(A, r, \xi) \quad (20)$$

where

$$K(A, r, \xi) = \left(\frac{1}{\xi}\right)^{\alpha-1} \frac{\vec{r} \left[\vec{r} \cdot \vec{A}(r, \omega) \right]}{r^4} E_{\alpha\beta}(-r/l), \quad (21)$$

and $I^\alpha K(A, r, \xi)$ is the fractional integral of kernel $K(A, r, \xi)$.

The fractional approach here is still justified by the argumentation that the particle described by this formalism can be thought as an pseudo-particle, as already commented, it would carry the information of the media and the kind of interactions implicit in the equation that describes his evolution. This pseudo-particle would be then “dressed” with information about media and interactions, and the solutions of the fractional equation are, like the Green functions in condensed matter physics, carrying additional information about iterations and the media. Then, even if the media is not fractal due to not so high energy regime, the fractional approach still makes sense to describe the evolutions of a pseudo-particle. This means that, in the fractional approach context, the particle interacts not as an isolated particle but as a pseudo-particle.

The phenomenology here is in some sence similar to that of an anomalous transport with different relaxation times, as in some non-Newtonian viscous systems, but with anomalous correlation, with some “memory effect” or heredity, that gives to each location a probability proportional to a power of distance from the source, it leads to non-local fractional effective theory. The α parameter of fractionality can than be used to create an alternative way to characterize superconductors.

VI. CONCLUDING REMARKS

In this work, by taking into account a non-differentiable space (coarse-grained), we have obtained a fractional linear London equation in terms of a fractional Laplacian with a sequential form of modified Riemann-Liouville fractional derivatives. We claim that the novelty of our work is the particular choice of formalisms with sequential modified fractional derivatives and an adequate chain rules, applied in the superconductivity area, leading to technique that creates a perspective to obtain solutions for other similar problems like Preisach hysteresis model [14, 15]. Our solutions are worked out by means of a complex transform in the

space variable that permits to change the fractional London Equation to an integer one with known Bessel like solutions. The MRL approach of fractional calculus seems to be more adequate to deal with problems that involve transformations in coordinates, since the chain and Leibniz rules are less complicated. Since we are choosing to work with non-differentiable coarse-grained space, no use of distributional generalized functions or fractional powers of operators, neither the maintenance of semi-group properties of exponents in the derivatives is made.

Our solution of fractional linear London model indicates that by means of a Meissner effect, the exclusion of the magnetic field is more effective in the vicinity of the boarder of a vortex than in the integer case but the residual field inside the vortex is greater than in the integer case one. This can also indicates that the fractional model could take into account influence of the contours of the grains to the characteristics of the Meissner effect in a Type II superconductor by means of a fractionality parameter α .

The results agrees with standard integer order in the convenient limit of $\alpha = 1$.

Also, based on a Chambers model for the anomalous skin effect, we have proposed a fractional Pippard-like model with non-local effects that can gives rises to an effective theory for superconductors and can be used to characterize and distinguish superconductors properties by the measurement of a fractionality parameter.

Acknowledgments

The author wishes to express his gratitude to Prof. Isaias G. de Oliveira (DEFIS-UFRRJ) by the discussions.

I am also very grateful to prof. Cresus F. L. Godinho (DEFIS-UFRRJ) by the text revision.

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